STOCHASTIC-DETERMINISTIC MODELING OF THE DEVELOPMENT OF HYDRODYNAMIC INSTABILITY IN FILTRATION OF MIXING FLUIDS

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The stochastic-deterministic approach to modeling of the Saffman–Taylor hydrodynamic instability in filtration of mixing fluids in a porous medium is considered. The numerical model is used to study the dynamics of development of the instability in displacement of a more viscous fluid by a less viscous one in uniformly and nonuniformly permeable media. Results of the modeling are compared to experimental data.

Introduction. Displacement of a more viscous fluid by a less viscous one from a porous medium or a Hele–Shaw cell can be accompanied by the development of the Saffman–Taylor hydrodynamic instability [1, 2]. The latter can occur in the case of both mixing [1, 3, 4] and immiscible fluids [2, 5]. The interest in this instability is associated with the problems of improvement of the efficiency of oil beds, conservation of underground-water resources, optimization of chemical-technology processes connected with the filtration of fluids through porous media, etc.

The present work is devoted to modeling of the Saffman–Taylor instability which arises in the case of displacement of mixing fluids in a porous medium. The experiments [3, 4, 6, 7] conducted with quasi-two-dimensional media showed that the development of the instability leads to the formation of stochastically bending and branching fingers of a displacing fluid in the region of a displaced fluid. The space-time characteristics of formation of the fingers are determined by the ratio of viscosities of the fluids and the relationship between convective transfer and hydrodispersion. In theoretical investigation of the Saffman–Taylor instability, both analytical and numerical methods are used. The former allow study of the conditions of occurrence of the instability [8, 9] and the asymptotic behavior of the fingers of a displacing fluid [10, 11]. Use of the latter makes it possible to study the dynamics of development of the instability in more detail. The growth of the fingers of a displacing fluid in the unstable mode of displacement was modeled using deterministic models based on numerical solution of the equations of filtration and mass transfer [6, 7, 12, 13]. Nonuniformities of the permeability of a medium [6, 7] or the initial disturbances of a fluid flow [12, 13] were used to initiate the instability. Spontaneous initiation of the instability due to the microinhomogeneity of the medium or fluctuations of the flow can be described by stochastic methods. For stochastic modeling of the development of the Saffman–Taylor instability use was made of randomly wandering particles [14, 15], random motion of the displacement front with a probability proportional to the pressure gradient [16–18], and random movement of the lines of equal concentration which is determined by the rate of change of the concentration [19]. However, the stochastic models [14–19] have limited possibilities for description of hydrodynamic instabilities which are caused by nonequilibrium mass and heat transfer, e.g., instability arising in nonisothermal filtration due to the dependence of viscosity on temperature, reaction-filtration instability associated with the dissolution of rock and increase in the permeability of the medium, etc. Modeling of the development of such instabilities requires a more extended description of mass and heat transfer compared to the stochastic models. In [20], a combined stochastic-deterministic method of modeling two-phase filtration, which allows one to describe the development of instability connected with nonequilibrium mass and/or heat transfer, was suggested. In the present work, the method proposed is developed and used to describe hydrodynamic instability in filtration of mixing fluids. We present the results of modeling of the development of the instability in displacement of a more viscous fluid by a less viscous one in uniformly and nonuniformly permeable media.

Formulation of the Problem. We consider the planned displacement of a more viscous fluid by a less viscous one from a horizontal bed. The fluids are assumed to be mixing and incompressible. Their viscosities are determined by the concentration of the dissolved component (thickener) which increases viscosity. The hydrodispersion arising as a result of the inhomogeneity of the porous-medium structure is described by a double-porosity approximation [21]. According to this approach, the pore space is divided into flow-through and stagnant pores occupied by the mobile and immobile parts of the fluid, respectively. Thus, the porosity of the medium m is the sum of the flow-through porosity m_1 and the stagnant porosity m_2. The rate of filtration $\mathbf{U}$ of the mobile part of the fluid is determined by the Darcy law [22]. The mobile and immobile parts of the fluid exchange dissolved substances. The density of the mass-exchange flow is taken to be proportional to the difference of the concentrations of the components in the mobile and immobile parts of the fluid. The system of equations which describes the dynamics of pressure P and the concentrations of the thickening component in the mobile C and immobile C^* parts of the fluid can be written in the form

$$\text{div } \mathbf{U} = 0 , \tag{1}$$

$$\mathbf{U} = - \frac{k}{\mu(C)} \text{ grad } P , \tag{2}$$

$$m_1 \frac{\partial C}{\partial t} = - \text{div}(C \cdot \mathbf{U}) - \alpha (C - C^*) , \tag{3}$$

$$m_2 \frac{\partial C^*}{\partial t} = \alpha (C - C^*) . \tag{4}$$

Fluid flow is considered in a rectangular region $D = (0 \leq x \leq L_x, \ 0 \leq y \leq L_y)$ with impermeable side boundaries $(\partial P / \partial y = 0; \ y = 0, L_y; \ 0 \leq x \leq L_x)$. The flow Q of a fluid with a zero concentration of the thickener $(C = 0)$ is set in the charging loop $(x = 0, \ 0 \leq y \leq L_y)$. Free sink $(P = 0)$ occurs in the discharge loop $(x = L_x, 0 \leq y \leq L_y)$. At the initial instant of time $(t = 0)$ the concentrations of the thickener in the flow-through and stagnant pores are equal $(C = C^* = C_0)$.

Formulation of the Stochastic-Deterministic Numerical Model. The system of equations (1)–(4) is discretized by uniform rectangular space and time grids with steps h and τ, respectively. Each node (i, j) of the space grid corresponds to an element of the medium with permeability $k_{i,j}$. The state of the node (i, j) at the time instant $t = n\tau$ is determined by the pressure $P_{i,j}^n$ and the concentrations of the thickener in the mobile $C_{i,j}^n$ and immobile $(C^*)_{i,j}^n$ parts of the fluid. Successive calculations of the pressure and the concentrations are made by the method of splitting by physical processes [23] at each time step n. First, we calculate the distribution of the pressure $P_{i,j}^n$ for concentrations corresponding to the previous time step. Then, on the basis of the found pressure distribution, we calculate new concentrations of the components in the flow-through pores $\widetilde{C}_{i,j}^n$ resulting from the convective motion of the fluid. Then, we determine changes in the concentrations due to the mass exchange between the mobile and immobile parts of the fluid and calculate the concentrations $C_{i,j}^n$ and $(C^*)_{i,j}^n$ at the nth time step.

The algorithm of calculation of the pressure distribution is based on the difference analog of Eqs. (1) and (2) written conservatively by a five-point template [24]:

$$\sum_{i',j'} U_{i,j,i',j'}^n = 0 , \tag{5}$$

$$U_{i,j,i',j'}^n = G_{i,j,i',j'}^{n-1} (P_{i,j}^n - P_{i',j'}^n) , \tag{6}$$

where $U_{i,j,i',j'}^n$ is the flow of the fluid between the nodes (i, j) and (i', j') and $G_{i,j,i',j'}^n$ is the hydraulic conductivity between the nodes (i, j) and (i', j'); summation in expression (5) is made over all the nodes neighboring the node (i, j). The hydraulic conductivity $G_{i,j,i',j'}^n$ is calculated from the formula
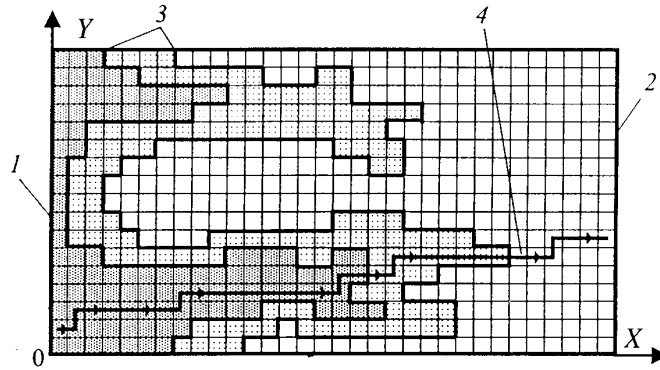
Fig. 1. Example of construction of a wandering streamline: 1) charging loop;
2) loop of discharge; 3) lines of equal concentration; 4) wandering streamline.

$$G_{i,j,i',j'}^{n} = 2\,(\mu\,(C_{i,j}^{n})/k_{i,j} + \mu\,(C_{i',j'}^{n})/k_{i',j'})^{-1}\,.$$

In solving the system of equations (5), (6) relative to the pressure $P_{i,j}^{n}$ by the method of iterations [23], use is made of the distributions of concentrations determined on the previous time layer.

The change in the concentration of the mobile part of the fluid due to convective mass transfer is calculated using stochastically wandering streamlines [20]. The streamlines begin on the charging loop and terminate on the discharge loop (Fig. 1). The probability $W_{i,j,i',j'}^{n}$ of motion of the streamline from the node (i, j) to the node (i', j') of the computational grid is taken to be proportional to the flow of the liquid phase $U_{i,j,i',j'}^{n}$ between the nodes, if the fluid flow is directed from the node (i, j) to the node (i', j'), and is equal to zero in the case of return flow

$$W_{i,j,i',j'}^{n} = \begin{cases} U_{i,j,i',j'}^{n}/Z_{i,j}^{n}, & U_{i,j,i',j'}^{n} > 0\,; \\ 0, & U_{i,j,i',j'}^{n} \le 0\,, \end{cases}$$

where $Z_{i,j}^{n}$ is the normalization factor ($Z_{i,j}^{n} = \sum\limits_{i',j'} U_{i,j,i',j'}^{n}$, summation is made over the nodes (i', j') neighboring (i, j) for which the condition $U_{i,j,i',j'}^{n} > 0$ holds). The motion of the streamline is initiated at one node of the charging loop with a probability proportional to the normal component of the flow. One streamline along which filtration of the fluid of volume $\Delta V = Q\tau$ occurs is constructed at each time step. In order to decrease numerical variance, the value of the time step τ is determined from the condition of equality of the volume ΔV to the volume of the mobile part of the fluid at one node of the grid:

$$\tau = m_{1}h^{2}/Q\,.$$

The concentration of the thickener in the mobile part of the fluid due to convective motion along the streamline is determined by the relation

$$\tilde{C}_{i,j}^{n} = C_{i',j'}^{n-1}\,,$$

where (i', j') is the node preceding the node (i, j) along the streamline. It is taken for all nodes not lying on the streamline that $\tilde{C}_{i,j}^{n} = C_{i,j}^{n-1}$.

The values of the concentration of the thickener $C_{i,j}^{n}$ and $(C^{*})_{i,j}^{n}$ on the time layer n, which arise as a result of mass exchange between the mobile and immobile parts of the fluid, are found from simultaneous solution of the difference analogs of the equation of mass exchange and the law of mass conservation:
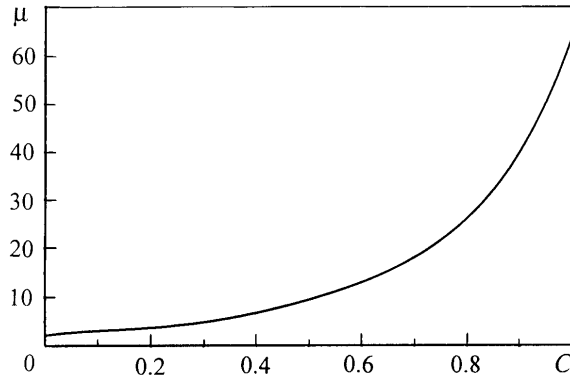
Fig. 2. Dependence of the viscosity μ (mPa·sec) on the concentration of glycerol C (rel. units) in the solution [6].
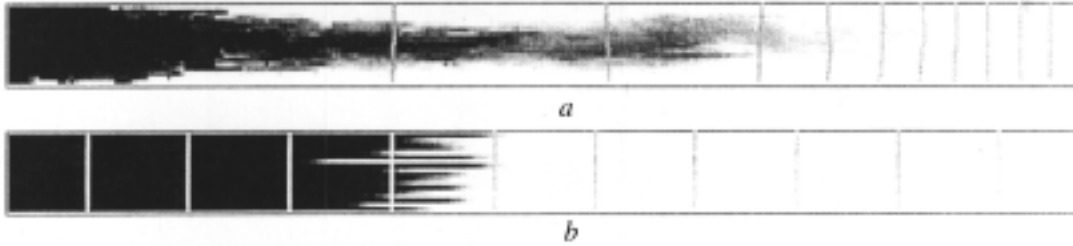

Fig. 3. Results of modeling of displacement at different ratios of viscosities: a) finger formation in displacement of the aqueous solution of glycerol (M = 30.6); b) displacement of the solution of a passive component (M = 1). The lines indicate isobars. The dark color corresponds to a lower concentration of the dissolved substance. The volume of the displaced fluid is 0.4 of the total volume of the pore space.

$$m_1 \frac{C_{i,j}^n - \tilde{C}_{i,j}^n}{\tau} = \alpha \left((C^*)_{i,j}^n - \tilde{C}_{i,j}^n \right), \quad m_1 \left(C_{i,j}^n - \tilde{C}_{i,j}^n \right) = -m_2 \left((C^*)_{i,j}^n - (C^*)_{i,j}^{n-1} \right).$$

Thus, in the model suggested, the stochastic interpretation of the Darcy law is used to describe convective mass transfer. The distribution of pressure and the mass exchange between the mobile and immobile parts of the fluid are calculated on the basis of corresponding deterministic regularities. The randomly wandering streamline used in the model greatly differs from the streamline used in the theory of filtration for fragmentation of a planned flow by streamlines [22]. The geometry of a classical streamline is constant, whereas a wandering streamline changes its position. The width of a classical streamline changes in movement along the axis of the line, while a wandering line has a constant width. At the same time, a classical streamline and the wandering streamline used in the model have important features in common. Their direction is determined by a pressure gradient and the statistically averaged position of the ensemble of wandering streamlines corresponds to the position of the classical streamline. Both lines begin on the charging loop and terminate on the loop of discharge.

Results and Discussion. The model was used to study the development of the hydrodynamic instability of displacement of mixing fluids in uniformly and nonuniformly permeable porous media. The conditions of modeling were selected so as to directly compare the results of the calculation to the data of experimental study of instability published in [6]. We considered the displacement of the aqueous solution of glycerol (μ = 64.32), used as a thickener, by the aqueous solution of potassium citrate of equal density (μ = 2.1). Thus, the ratio M of the viscosity of the displaced fluid to the viscosity of the displacing one was 30.6. The dependence of the solution viscosity on the relative concentration of glycerol ($\mu(C) = 2.1 + 17.037C - 108.29C^2 + 418.28C^3 - 581.15C^4 + 316.34C^5$) is presented in Fig. 2 [6]. The unit concentration of glycerol corresponds to the displaced solution (C_0 = 1). The dimensions of the region of modeling L_x = 0.8 m and L_y = 0.06 m, flow rate of the displacing fluid $Q = 4.8 \cdot 10^{-7}$ m^2/sec, porosity m = 0.38,
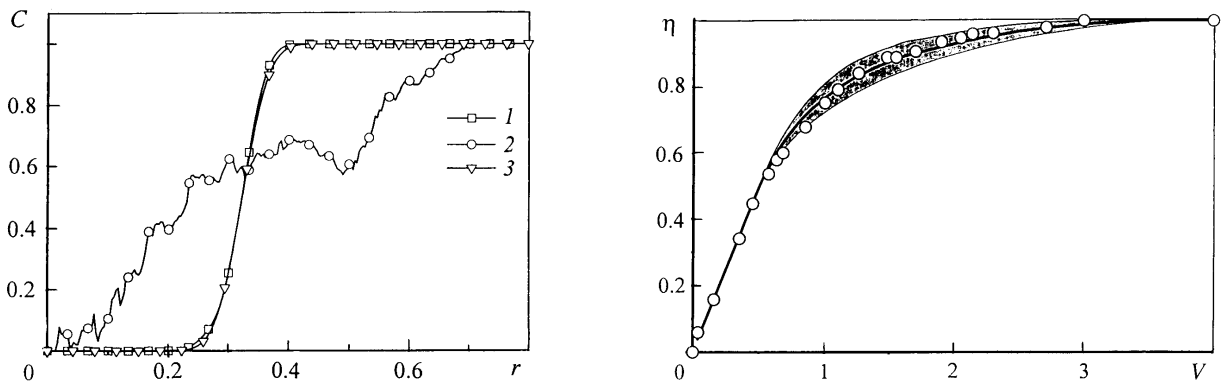
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Fig. 4. Dependence of the averaged concentration C (rel. units) on the distance r (m) to the charging loop: 1) passive component in the stable mode of displacement; 2) glycerol in the unstable mode of displacement; 3) result of a one-dimensional analytical calculation. The volume of the displaced fluid is 0.4 of the total volume of the pore space.

Fig. 5. Dependence of the fraction η of the displaced glycerol on the volume V of the displacing solution (fraction of the total pore space).

and permeability $k = 6.5 \cdot 10^{-12}\,\mathrm{m}^2$ corresponded to the conditions of [6]. The coefficient of mass exchange $\alpha = 0.02\,\mathrm{sec}^{-1}$, flow porosity $m_1 = 0.27$, and stagnant porosity $m_2 = 0.11$ were determined on the basis of comparison of the results of modeling to the experimental data [6]. Calculations were made on a 240×18-node grid.

A typical result of modeling of the growth of a finger of a displacing fluid in the solution of glycerol as a result of the development of hydrodynamic instability is presented in Fig. 3a. For comparison, the pattern of displacement obtained for the same values of the viscosities of the displacing and displaced fluids (in the displaced fluid, glycerol is replaced by a passive component not affecting viscosity) is shown in Fig. 3b. The observed nonuniformity of the displacement front corresponds to hydraulic dispersion arising as a result of the inhomogeneity of the velocity field of the fluid in the pore space. A similar pattern of the displacement front for the same viscosities of the fluids was obtained in [25]. The profiles of the averaged concentration of glycerol and the passive component are presented in Fig. 4. The result of the analytical solution of the one-dimensional equation of convection-dispersion transfer [22], obtained for the same conditions of filtration and the coefficient of dispersion $D = 1.5 \cdot 10^{-9}\,\mathrm{m}^2/\mathrm{sec}$, is shown for comparison. It is seen that in the stable mode of displacement the profile of the averaged concentration of the passive component coincides with the analytical solution. The development of fingers in unstable displacement facilitates the motion of the displacing fluid and results in the broadening of the displacement front. To make quantitative comparison of the results of the modeling and the experiment we studied the dependence of the volume fraction of the displaced glycerol on the volume of the displacing fluid. In Fig. 5, the solid line corresponds to the result of averaging over ten numerical experiments (the value of the standard deviation is shown in gray) and the points correspond to experimental data [6]. The rather good agreement between the results of the modeling and the experimental data [6] indicates the adequacy of the model suggested.

The dynamics of development of fingers and the effect of inhomogeneous permeability of the medium were studied in a shorter and more extended region of modeling ($L_x = 0.4$ m, $L_y = 0.12$, 120×40-node computational grid). A typical picture of the development of instability is presented in Fig. 6. Several thin fingers of the displacing fluid are formed at the initial stage. During the growth, the fingers branch randomly and bend. The redistribution of pressure leads to the origination of the effect of screening and suppression of the growth of retarding fingers. The development of the dominant fingers is accompanied by their broadening. The picture of development of the instability obtained in modeling is in agreement with the results of experimental study of the displacement of mixing fluids in a porous medium [3, 4, 6, 7]. The inhomogeneities of the permeability change the distributions of the pressure and the fluid flow in the medium. The results of the modeling show that in this case the position of the dominant fingers of the displacing fluid corresponds to sites with the highest pressure gradient. Moreover, the presence of the inhomogeneity leads to the acceleration of the development of instability. Figure 7 shows examples of the development of in-
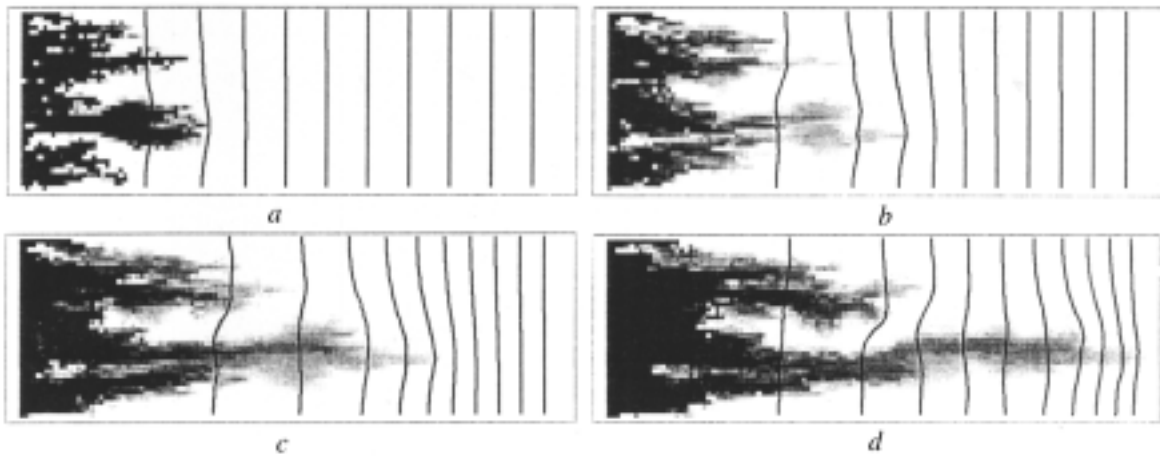
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Fig. 6. Result of modeling of the development of hydrodynamic instability in displacement of the aqueous solution of glycerol ($M = 30.6$). The volume of the displaced fluid (fraction of the total volume of the pore space) is: 1) 0.1, b) 0.2, c) 0.3, and d) 0.4.
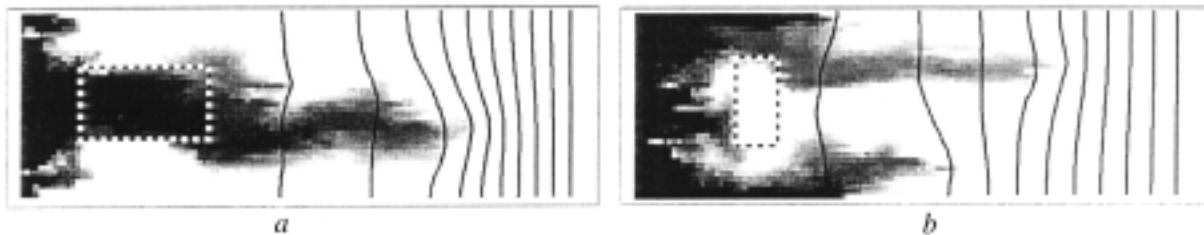

Fig. 7. Result of modeling of the development of hydrodynamic instability in a nonuniformly permeable medium: a) high-permeability inclusion ($k_{incl}/k = 10$); b) low-permeability inclusion ($k_{incl}/k = 0.1$). The position of the inclusion is shown dashed.

stability in the presence of rectangular inclusions with permeability k_{incl} (decreased, $k_{incl}/k = 0.1$, and increased, $k_{incl}/k = 10$, compared to the main medium) in the medium. Thus, the introduction of the inhomogeneity of the permeability of the porous medium makes it possible to affect the development of instability.

Conclusions. The suggested stochastic-deterministic model adequately describes the main laws governing the development of hydrodynamic instability in the case of displacement of a fluid containing a substance with increased viscosity by a less viscous fluid in a porous medium. The results of modeling the growth of fingers of the displacing fluid due to the development of instability agree with the data of the laboratory experiments conducted with quasi-two-dimensional media. In a nonuniformly permeable medium, the position of the fingers is determined by sites with the highest gradient of pressure which allows control of the growth of the fingers of the displacing fluid by high- and low-permeability inclusions. The results of the work can be used to predict the propagation of contamination in underground water to regenerate porous catalysts in the processes of chemical technology, to develop chromatographic methods of analysis, etc.

NOTATION

α, mass-transfer coefficient; C and C^*, concentrations of the thickener in the mobile and immobile parts of the fluid; G, hydraulic conductivity; h, period of the space grid; k, permeability of the medium; k_{incl}, permeability of the inclusion; L_x and L_y, size of the region of modeling along the coordinate axes X and Y; μ, viscosity; m, porosity

of the medium; P, hydrostatic pressure; Q, fluid flow through the charging loop; t, time; τ, period of the time grid; V, volume; x, y, Cartesian coordinates; W, probability.

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